

# Wild ramification determines the characteristic cycle

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## Abstract

Constructible complexes have the same characteristic cycle if they have the same wild ramification, even if the characteristics of the coefficients fields are different.

The characteristic cycle  $CC\mathcal{F}$  of a constructible complex  $\mathcal{F}$  on a smooth variety  $X$  over a perfect field  $k$  is defined in [S], as a cycle on the cotangent bundle  $T^*X$  supported on the singular support  $SS\mathcal{F}$  defined by Beilinson in [B]. The characteristic cycle is characterized by the Milnor formula recalled in Theorem 1.3 computing the total dimension of the space of vanishing cycles.

We show that constructible complexes have the same characteristic cycle if they have the *same wild ramification*. This terminology will be defined in Definition 5.1 in the text.

**Theorem 0.1.** *Let  $X$  be a smooth scheme over a perfect field  $k$  and let  $\Lambda$  and  $\Lambda'$  be finite fields of characteristic invertible in  $k$ . Let  $\mathcal{F}$  and  $\mathcal{F}'$  be constructible complexes of  $\Lambda$ -modules and of  $\Lambda'$ -modules on  $X$  respectively. If  $\mathcal{F}$  and  $\mathcal{F}'$  have the same wild ramification, we have*

$$(0.1) \quad CC\mathcal{F} = CC\mathcal{F}'.$$

A special case where  $\Lambda = \Lambda'$  is proved in the thesis of the second named author [Y, Theorem 7.25]. Theorem 0.1 is a refinement of and is deduced from the following equality of Euler characteristic.

**Proposition 0.2** (cf. [I, Théorème 2.1]). *Let  $X$  be a separated scheme of finite type over an algebraically closed field  $k$  and let  $\Lambda$  and  $\Lambda'$  be finite fields of characteristic invertible in  $k$ . Let  $\mathcal{F}$  and  $\mathcal{F}'$  be constructible complexes of  $\Lambda$ -modules and of  $\Lambda'$ -modules on  $X$  respectively. If  $\mathcal{F}$  and  $\mathcal{F}'$  have the same wild ramification, we have*

$$(0.2) \quad \chi_c(X, \mathcal{F}) = \chi_c(X, \mathcal{F}').$$

A special case where  $\Lambda = \Lambda'$  is proved in [I, Théorème 2.1].

To deduce Theorem 0.1 from Proposition 0.2, we take a morphism to a curve and use the Grothendieck-Ogg-Shafarevich formula to recover the total dimension of the space of vanishing cycles appearing in the characterization of characteristic cycle from the Euler-Poincaré characteristic.

We briefly recall the definition and properties of singular support and characteristic cycle in Section 1. As preliminaries of proof of Theorem 0.1, we prove the existence of a good pencil in Section 2. We show that the characteristic cycle of a sheaf is determined by the Euler-Poincaré characteristics of its pull-backs using the existence of a good pencil

in Section 3. Finally, we prove Theorem 0.1 after defining the condition for constructible complexes to have the same wild ramification in Section 4.

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## 1 Characteristic cycle

We briefly recall the definition of characteristic cycle. We refer to [S] for more detail. For a smooth scheme  $X$  over a field  $k$ , let  $T^*X = \text{Spec } S^\bullet \Omega_X^{1\vee}$  be the cotangent bundle of  $X$  and let  $T_X^*X$  denote the zero section. A morphism  $f: X \rightarrow Y$  of smooth schemes over  $k$  induces a linear mapping  $df: X \times_Y T^*Y \rightarrow T^*X$  of vector bundles on  $X$ . We say that a closed subset  $C$  of a vector bundle is *conical* if  $C$  is stable under the action by the multiplicative group.

**Definition 1.1** ([B, 1.2]). Let  $X$  be a smooth scheme over a field  $k$  and let  $C \subset T^*X$  be a closed conical subset.

1. Let  $h: W \rightarrow X$  be a morphism of smooth schemes over  $k$ . We say that  $h$  is *C-transversal* if we have

$$dh^{-1}(T_W^*W) \cap h^*C \subset W \times_X T_X^*X,$$

where  $h^*C = W \times_X C$ .

For a  $C$ -transversal morphism  $h: W \rightarrow X$ , we define a closed conical subset  $h^\circ C \subset T^*W$  to be the image of  $h^*C \subset W \times_X T^*X$  by the morphism  $dh: W \times_X T^*X \rightarrow T^*W$ .

2. Let  $f: X \rightarrow Y$  be a morphism of smooth schemes over  $k$ . We say that  $f$  is *C-transversal* if we have

$$df^{-1}(C) \subset X \times_Y T_Y^*Y.$$

3. Let  $h: W \rightarrow X$  and  $f: W \rightarrow Y$  be morphisms of smooth schemes over  $k$ . We say that the pair  $(h, f)$  is *C-transversal* if  $h$  is  $C$ -transversal and if  $f$  is  $h^\circ C$ -transversal.

4. Let  $j: U \rightarrow X$  be an étale morphism,  $f: U \rightarrow Y$  a morphism over  $k$  to a smooth curve over  $k$ , and  $u \in U$  a closed point. We say that  $u$  is an *isolated characteristic point* with respect to  $C$  if the pair  $(j, f)$  is not  $C$ -transversal and its restriction to  $U - \{u\}$  is  $C$ -transversal.

Let  $\Lambda$  be a finite field of characteristic  $\ell$  invertible in  $k$ . We say that a complex  $\mathcal{F}$  of étale sheaves of  $\Lambda$ -modules on  $X$  is *constructible* if the cohomology sheaf  $\mathcal{H}^q(\mathcal{F})$  is constructible for every  $q$  and if  $\mathcal{H}^q(\mathcal{F}) = 0$  except finitely many  $q$ .

**Definition 1.2** ([B, 1.3]). Let  $X$  be a smooth scheme over a field  $k$  and let  $\Lambda$  be a finite field of characteristic  $\ell$  invertible in  $k$ . Let  $\mathcal{F}$  be a constructible complex of  $\Lambda$ -modules on  $X$ .

1. Let  $C \subset T^*X$  be a closed conical subset. We say that  $\mathcal{F}$  is *micro-supported* on  $C$  if for every  $C$ -transversal pair  $(h, f)$  of morphisms  $h: W \rightarrow X$  and  $f: W \rightarrow Y$  of smooth schemes over  $k$ , the morphism  $f$  is locally acyclic relatively to  $h^*\mathcal{F}$ .

2. The *singular support*  $SS\mathcal{F}$  of  $\mathcal{F}$  is the smallest closed conical subset  $C$  of  $T^*X$  on which  $\mathcal{F}$  is micro-supported.

By [B, Theorem 1.3], the singular support exists for every constructible complex of  $\Lambda$ -modules. Further, if  $X$  is equidimensional of dimension  $n$ , the singular support is equidimensional of dimension  $n$ .

**Theorem 1.3** (Milnor formula, [S, Theorem 5.9, Theorem 5.18]). *Let  $X$  be a smooth scheme equidimensional of dimension  $n$  over a perfect field  $k$  and let  $\Lambda$  be a finite field of characteristic  $\ell$  invertible in  $k$ . Let  $\mathcal{F}$  be a constructible complex of  $\Lambda$ -modules on  $X$  and  $C \subset T^*X$  a closed conical subset. Assume that  $\mathcal{F}$  is micro-supported on  $C$  and that every irreducible components  $C_a$  of  $C = \bigcup_a C_a$  is of dimension  $n$ .*

*Then, there exists a unique  $\mathbf{Z}$ -linear combination  $A = \sum_a m_a C_a$  satisfying the following condition: Let  $(j, f)$  be the pair of an étale morphism  $j: U \rightarrow X$  and a morphism  $f: U \rightarrow Y$  over  $k$  to a smooth curve over  $k$ . Let  $u \in U$  be a closed point such that  $u$  is at most an isolated characteristic point of  $f$  with respect to  $C$ . Then we have*

$$(1.1) \quad -\dim_{\text{tot}} \phi_u(j^*\mathcal{F}, f) = (j^*A, df)_{T^*U, u}.$$

*Further  $A$  is independent of  $C$  on which  $\mathcal{F}$  is micro-supported.*

In (1.1), the left hand side denotes the minus of the total dimension of the stalk  $\phi_u(j^*\mathcal{F}, f)$  at  $u$  of the complex of vanishing cycles. The total dimension  $\dim_{\text{tot}}$  is defined as the sum of the dimension and the Swan conductor. The right hand side denotes the intersection number supported on the fiber of  $u$  of the pull-back  $j^*A$  with the section  $df$  defined to be the pull-back of  $dt$  for a local coordinate  $t$  of  $Y$  at  $f(u)$ .

**Definition 1.4** ([S, Definition 5.10]). Let  $X$  be a smooth scheme over a perfect field  $k$  and let  $\Lambda$  be a finite field of characteristic  $\ell$  invertible in  $k$ . Let  $\mathcal{F}$  be a constructible complex of  $\Lambda$ -modules on  $X$ . We define the *characteristic cycle*  $CC\mathcal{F}$  of  $\mathcal{F}$  to be  $A = \sum_a m_a C_a$  in Theorem 1.3.

For  $\mathbf{Z}_\ell$ -coefficient or  $\mathbf{Q}_\ell$ -coefficient, the characteristic cycle is defined by taking the reduction modulo  $\ell$ . Theorem 1.3 implies the following additivity of characteristic cycles. For a distinguished triangle  $\mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow$  of constructible complexes of  $\Lambda$ -modules, we have

$$(1.2) \quad CC\mathcal{F} = CC\mathcal{F}' + CC\mathcal{F}''.$$

## 2 Existence of good pencil

Let  $X$  be a smooth projective scheme over an algebraically closed field  $k$ . Let  $\mathcal{L}$  be a very ample invertible  $\mathcal{O}_X$ -module and let  $E \subset \Gamma(X, \mathcal{L})$  be a sub  $k$ -vector space defining a closed immersion  $i: X \rightarrow \mathbf{P} = \mathbf{P}(E^\vee) = \text{Proj}_k S^\bullet E$ . The dual projective space  $\mathbf{P}^\vee = \mathbf{P}(E)$  parametrizes hyperplanes in  $\mathbf{P}$ . We identify the universal hyperplane  $Q = \{(x, H) \mid x \in H\} \subset \mathbf{P} \times_k \mathbf{P}^\vee$  with the covariant projective space bundle  $\mathbf{P}(T^*\mathbf{P})$  as in the beginning of

[S, Subsection 3.2]. We also identify the universal family of hyperplane sections  $X \times_{\mathbf{P}} Q$  with  $\mathbf{P}(X \times_{\mathbf{P}} T^*\mathbf{P})$ .

For a line  $L \subset \mathbf{P}^\vee$ , let  $A_L$  denote the axis  $\bigcap_{t \in L} H_t$  of  $L$ . Define  $p_L: X_L = \{(x, H_t) \mid x \in X \cap H_t, t \in L\} \rightarrow L$  by the cartesian diagram

$$(2.1) \quad \begin{array}{ccc} X_L & \longrightarrow & X \times_{\mathbf{P}} Q \\ p_L \downarrow & & \downarrow \\ L & \longrightarrow & \mathbf{P}^\vee. \end{array}$$

The projection  $\pi_L: X_L \rightarrow X$  induces an isomorphism on the complement  $X_L^\circ = X - X \cap A_L$ . Let  $p_L^\circ: X_L^\circ \rightarrow L$  be the restriction of  $p_L$ . We note that if  $A_L$  meets  $X$  transversally then  $\pi_L: X_L \rightarrow X$  is the blow-up of  $X$  along  $X \cap A_L$  and hence  $X_L$  is smooth over  $k$ .

Let  $C \subset T^*X$  be a closed conical subset and let  $\tilde{C}$  be the inverse image by the surjection  $di: X \times_{\mathbf{P}} T^*\mathbf{P} \rightarrow T^*X$ . We consider the following conditions.

(E) For every pair  $(u, v)$  of distinct closed points of  $X$ , the restriction

$$E \subset \Gamma(X, \mathcal{L}) \rightarrow \mathcal{L}_u/\mathfrak{m}_u^2 \mathcal{L}_u \oplus \mathcal{L}_v/\mathfrak{m}_v^2 \mathcal{L}_v$$

is surjective.

(C) For every irreducible component  $C_a$  of  $C$ , the inverse image  $\tilde{C}_a \subset \tilde{C}$  of  $C_a$  by the surjection  $di: X \times_{\mathbf{P}} T^*\mathbf{P} \rightarrow T^*X$  is not contained in the 0-section  $X \times_{\mathbf{P}} T^*\mathbf{P}$ .

If every irreducible component of  $C_a$  has the same dimension as that of  $X$ , the condition (C) is satisfied unless  $X = \mathbf{P}$ . Hence, by [S, Lemma 3.19], after replacing  $\mathcal{L}$  and  $E$  by  $\mathcal{L}^{\otimes n}$  and the image  $E^{(n)}$  of  $E^{\otimes n} \rightarrow \Gamma(X, \mathcal{L}^{\otimes n})$  for  $n \geq 3$  if necessary, the condition (E) and (C) are satisfied if every irreducible component  $C_a$  of  $C = \bigcup_a C_a$  is of dimension  $\dim X$ . For each irreducible component  $C_a$  of  $C$ , we regard the projectivization  $\mathbf{P}(\tilde{C}_a)$  of  $\tilde{C}_a \subset \tilde{C}$  as a closed subset of  $\mathbf{P}(\tilde{C}) \subset \mathbf{P}(X \times_{\mathbf{P}} T^*\mathbf{P}) = X \times_{\mathbf{P}} Q$ .

**Lemma 2.1.** *Assume that the axis  $A_L$  meets  $X$  transversally and that the immersion  $Z = X \cap A_L \rightarrow X$  is  $C$ -transversal.*

1. *The blow-up  $\pi_L: X_L \rightarrow X$  is  $C$ -transversal.*

2. *The intersection  $X_L \cap \mathbf{P}(\tilde{C})$  in  $\mathbf{P}(X \times_{\mathbf{P}} T^*\mathbf{P}) = X \times_{\mathbf{P}} Q$  is the smallest closed subset outside of which the projection  $p_L: X_L \rightarrow L$  is  $\pi_L^\circ C$ -transversal.*

*Proof.* 1. We consider the commutative diagram

$$\begin{array}{ccc} D \times_X T^*X & \longrightarrow & D \times_Z T^*Z \\ \downarrow & & \downarrow \\ D \times_{X_L} T^*X_L & \longrightarrow & T^*D \end{array}$$

of morphisms of vector bundles on the exceptional divisor  $D = Z \times_X X_L$ . Since  $D$  is smooth over  $Z$ , the right vertical arrow is injective. Hence, the kernel of the left vertical arrow is a subset of the kernel of the upper horizontal arrow. Thus, if the immersion  $Z \rightarrow X$  is  $C$ -transversal then  $\pi: X_L \rightarrow X$  is  $C$ -transversal.

2. Since  $p: X \times_{\mathbf{P}} Q \rightarrow X$  is smooth, the immersion  $X_L \rightarrow X \times_{\mathbf{P}} Q$  is  $p^\circ C$ -transversal by [S, Lemma 3.4.3]. Hence, by [S, Lemma 3.9.1] applied to the cartesian diagram

$$\begin{array}{ccc} X \times_{\mathbf{P}} Q & \longleftarrow & X_L \\ p^\vee \downarrow & & \downarrow p_L \\ \mathbf{P}^\vee & \longleftarrow & L \end{array}$$

and by [S, Lemma 3.10], the complement  $X_L - X_L \cap \mathbf{P}(\tilde{C})$  is the largest open subset where  $p_L: X_L \rightarrow L$  is  $\pi_L^\circ C$ -transversal.  $\square$

To show the existence of good pencil, we consider the universal family of pencils. Assume that  $X$  is projective smooth and let  $E \subset \Gamma(X, \mathcal{L})$  be a subspace of finite dimension defining a closed immersion  $X \rightarrow \mathbf{P} = \mathbf{P}(E^\vee)$  as above. We identify the Grassmannian variety  $\mathbf{G} = \text{Gr}(2, E)$  parametrizing subspaces of dimension 2 of  $E$  with the Grassmannian variety  $\mathbf{G} = \text{Gr}(1, \mathbf{P}^\vee)$  parametrizing lines in  $\mathbf{P}^\vee$ . The universal family  $\mathbf{A} \subset \mathbf{P} \times \mathbf{G}$  of linear subspace of codimension 2 of  $\mathbf{P} = \mathbf{P}(E^\vee)$  consists of pairs  $(x, L)$  of a point  $x$  of the axis  $A_L \subset \mathbf{P}$  of a line  $L \subset \mathbf{P}^\vee$ . Similarly as  $Q = \{(x, H) \in \mathbf{P} \times \mathbf{P}^\vee \mid x \in H\}$  is identified with  $\mathbf{P}(T^*\mathbf{P})$ , the scheme  $\mathbf{A}$  is identified with the Grassmannian bundle  $\text{Gr}(2, T^*\mathbf{P})$  parametrizing rank 2 subbundles of  $T^*\mathbf{P}$  by the injection  $T^*\mathbf{P}(1) \rightarrow E \times \mathbf{P}$ . The intersection  $X \times_{\mathbf{P}} \mathbf{A} = (X \times \mathbf{G}) \cap \mathbf{A}$  is canonically identified with the bundle  $\text{Gr}(2, X \times_{\mathbf{P}} T^*\mathbf{P})$  of Grassmannian varieties.

Let  $\mathbf{D} = \{(H, L) \in \mathbf{P}^\vee \times \mathbf{G} \mid H \in L\} \subset \mathbf{P}^\vee \times \mathbf{G}$  be the universal line over  $\mathbf{G}$ . We canonically identify the fiber product  $X \times_{\mathbf{P}} \mathbf{A} \times_{\mathbf{G}} \mathbf{D}$  with the flag bundle  $\text{Fl}(1, 2, X \times_{\mathbf{P}} T^*\mathbf{P})$  parametrizing pairs of sub line bundles and rank 2 subbundles of  $X \times_{\mathbf{P}} T^*\mathbf{P}$  with inclusions. We consider the commutative diagram

$$(2.2) \quad \begin{array}{ccccc} X \times_{\mathbf{P}} Q & \longleftarrow & X \times_{\mathbf{P}} \mathbf{A} \times_{\mathbf{G}} \mathbf{D} & \longrightarrow & X \times_{\mathbf{P}} \mathbf{A} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{P}^\vee & \longleftarrow & \mathbf{D} & \longrightarrow & \mathbf{G} \end{array}$$

defined as

$$(2.3) \quad \begin{array}{ccccc} \text{Gr}(1, X \times_{\mathbf{P}} T^*\mathbf{P}) & \longleftarrow & \text{Fl}(1, 2, X \times_{\mathbf{P}} T^*\mathbf{P}) & \longrightarrow & \text{Gr}(2, X \times_{\mathbf{P}} T^*\mathbf{P}) \\ \downarrow & & \downarrow & & \downarrow \\ \text{Gr}(1, E) & \longleftarrow & \text{Fl}(1, 2, E) & \longrightarrow & \text{Gr}(2, E). \end{array}$$

The horizontal arrows are forgetful morphisms and the vertical arrows are induced by the canonical injection  $\Omega_{\mathbf{P}}^1(1) \rightarrow E \otimes \mathcal{O}_{\mathbf{P}}$ . The right square is cartesian.

Let  $C \subset T^*X$  be a closed conical subset. Define a closed subset

$$(2.4) \quad \mathbf{R}(\tilde{C}) \subset X \times_{\mathbf{P}} \mathbf{A} = \text{Gr}(2, X \times_{\mathbf{P}} T^*\mathbf{P}) \subset X \times \mathbf{G}$$

to be the subset consisting of  $(x, V)$  such that the intersection  $V \cap (x \times_X \tilde{C}) \subset T^*\mathbf{P}$  is not a subset of 0. We also define a closed subset

$$(2.5) \quad \mathbf{Q}(\tilde{C}) \subset X \times_{\mathbf{P}} \mathbf{A} \times_{\mathbf{G}} \mathbf{D}$$

to be the inverse image of  $\mathbf{P}(\tilde{C}) \subset X \times_{\mathbf{P}} Q$  by the upper left horizontal arrow in (2.2). The subset  $\mathbf{R}(\tilde{C}) \subset X \times_{\mathbf{P}} \mathbf{A}$  is the image of  $\mathbf{Q}(\tilde{C})$  by the upper right arrow  $X \times_{\mathbf{P}} \mathbf{A} \times_{\mathbf{G}} \mathbf{D} \rightarrow X \times_{\mathbf{P}} \mathbf{A}$  of (2.2).

**Lemma 2.2.** *Let  $C \subset T^*X$  be a conical closed subset.*

1. *The complement  $X \times_{\mathbf{P}} \mathbf{A} - \mathbf{R}(\tilde{C})$  is the largest open subset where the pair  $(p, p')$  of  $p: X \times_{\mathbf{P}} \mathbf{A} \rightarrow X$  and  $p': X \times_{\mathbf{P}} \mathbf{A} \rightarrow \mathbf{G}$  is  $C$ -transversal.*

2. *For a line  $L \subset \mathbf{P}^\vee$  such that  $A_L$  meets  $X$  transversally, the following conditions are equivalent:*

(1) *The immersion  $Z = X \cap A_L \rightarrow X$  is  $C$ -transversal.*

(2) *The point of  $\mathbf{G}$  corresponding to  $L$  is not contained in the image of  $\mathbf{R}(\tilde{C}) \subset X \times_{\mathbf{P}} \mathbf{A}$  by  $X \times_{\mathbf{P}} \mathbf{A} \rightarrow \mathbf{G}$ .*

(3) *The pair  $X \leftarrow X_L \rightarrow L$  is  $C$ -transversal on a neighborhood of  $\pi_L^{-1}(Z) = Z \times L \subset X_L$ .*

*Proof.* 1. By [S, Lemma 3.6.9], the largest open subset  $U \subset X \times_{\mathbf{P}} \mathbf{A}$  where the pair  $(p, p')$  is  $C$ -transversal equals the largest open subset where  $(p, p'): X \times_{\mathbf{P}} \mathbf{A} \rightarrow X \times \mathbf{G}$  is  $C \times T^*\mathbf{G}$ -transversal. The kernel  $\text{Ker}((X \times_{\mathbf{P}} \mathbf{A}) \times_X T^*X \oplus (X \times_{\mathbf{P}} \mathbf{A}) \times_{\mathbf{G}} T^*\mathbf{G} \rightarrow T^*(X \times_{\mathbf{P}} \mathbf{A}))$  is canonically identified with the conormal bundle  $T_{X \times_{\mathbf{P}} \mathbf{A}}^*(X \times \mathbf{G})$  and further with the restriction of the universal sub vector bundle of rank 2 of  $T^*\mathbf{P}$  on  $\mathbf{A} = \text{Gr}(2, T^*\mathbf{P})$ . Hence,  $U$  is the complement of  $\mathbf{R}(\tilde{C})$ .

2. (1) $\Leftrightarrow$ (2): Since  $p: X \times_{\mathbf{P}} \mathbf{A} \rightarrow X$  is smooth, the condition (1) is equivalent to that the immersion  $Z \rightarrow X \times_{\mathbf{P}} \mathbf{A}$  is  $p^\circ C$ -transversal by [S, Lemma 3.4.1] and [S, Lemma 3.4.3]. We consider the cartesian diagram

$$(2.6) \quad \begin{array}{ccc} X \times_{\mathbf{P}} \mathbf{A} & \longleftarrow & Z \\ \downarrow & & \downarrow \\ \mathbf{G} & \longleftarrow & \{L\}. \end{array}$$

Since the right vertical arrow  $Z \rightarrow \{L\}$  in (2.6) is smooth by the assumption that the axis  $A_L$  meets  $X$  transversely, it is further equivalent to that the left vertical arrow  $X \times_{\mathbf{P}} \mathbf{A} \rightarrow \mathbf{G}$  is  $p^\circ C$ -transversal on a neighborhood of  $Z$  by [S, Lemma 3.6.1] and [S, Lemma 3.9.1]. Thus the assertion follows from 1.

(2) $\Leftrightarrow$ (3): The condition (2) means that  $Z \times L \subset X_L$  does not meet  $X_L \cap \mathbf{P}(\tilde{C})$ . This is equivalent to the condition (3) by Lemma 2.1.2.  $\square$

**Lemma 2.3.** *Let  $X$  be a smooth projective scheme equidimensional of dimension  $n$  over an algebraically closed field  $k$ . Let  $C \subset T^*X$  be a closed conical subset equidimensional of dimension  $n$ . Let  $\mathcal{L}$  be a very ample invertible  $\mathcal{O}_X$ -module and  $E \subset \Gamma(X, \mathcal{L})$  a sub  $k$ -vector space satisfying the condition (E) and (C). Let  $\mathbf{G} = \text{Gr}(1, \mathbf{P}^\vee)$  be the Grassmannian variety parameterizing lines in  $\mathbf{P}^\vee$ . Then, there exists a dense open subset  $U \subset \mathbf{G}$  consisting of lines  $L \subset \mathbf{P}^\vee$  satisfying the following conditions (1)–(7):*

(1) *The axis  $A_L$  meets  $X$  transversally and the immersion  $X \cap A_L \rightarrow X$  is  $C$ -transversal.*

(2) *The blow-up  $\pi_L: X_L \rightarrow X$  is  $C$ -transversal.*

(3) *The morphism  $p_L: X_L \rightarrow L$  has at most isolated characteristic points with respect to  $\pi_L^\circ C$ .*

(4) *For every closed point  $y$  of  $L$ , there exists at most one point  $x$  on the fiber  $X_{L,y}$  at  $y$  that is an isolated characteristic point of  $p_L: X_L \rightarrow L$ .*

(5) *No isolated characteristic point of  $p_L$  is contained in the inverse image by  $\pi_L: X_L \rightarrow X$  of the intersection  $X \cap A_L$ .*



- (6) For every irreducible component  $C_a$  of  $C$ , the intersection  $X_L \cap \mathbf{P}(\tilde{C}_a)$  is non-empty.  
(7) For every pair of irreducible components  $C_a \neq C_b$  of  $C$ , the intersection  $X_L \cap \mathbf{P}(\tilde{C}_a) \cap \mathbf{P}(\tilde{C}_b)$  is empty.

*Proof.* By Bertini, the lines  $L \subset \mathbf{P}^\vee$  such that the axis  $A_L \subset \mathbf{P}$  intersects  $X$  transversely form a dense open subset  $U_0 \subset \mathbf{G}$ . We show that the image of  $\mathbf{R}(\tilde{C})$  by  $p': X \times_{\mathbf{P}} \mathbf{A} \rightarrow \mathbf{G}$  is not dense. Since  $\mathbf{P}(\tilde{C}) \subset X \times_{\mathbf{P}} Q$  is of codimension  $n$ , its inverse image  $\mathbf{Q}(\tilde{C}) \subset X \times_{\mathbf{P}} Q \times_{\mathbf{G}} \mathbf{D}$  is also of codimension  $n$ . Since  $\mathbf{D}$  is a  $\mathbf{P}^1$ -bundle over  $\mathbf{G}$  the subset  $\mathbf{R}(\tilde{C}) \subset X \times_{\mathbf{P}} \mathbf{A}$  is of codimension  $\geq n - 1$ . Since  $\dim X \times_{\mathbf{P}} \mathbf{A} = \dim \mathbf{G} + n - 2$ , the image of  $\mathbf{R}(\tilde{C})$  by  $X \times_{\mathbf{P}} \mathbf{A} \rightarrow \mathbf{G}$  is not dense, as claimed.

Let  $L$  be a line corresponding to a point of  $U_1 = U_0 - (U_0 \cap p'(\mathbf{R}(\tilde{C}))) \subset \mathbf{G}$ . Then, the condition (1) is satisfied by Lemma 2.2.2 (2) $\Rightarrow$ (1). Hence the condition (2) is satisfied by Lemma 2.1.1. By Lemma 2.1.2, the condition (3) is satisfied if and only if the intersection  $X_L \cap \mathbf{P}(\tilde{C})$  is finite. Further the condition (4) is satisfied if and only if the restriction  $X_L \cap \mathbf{P}(\tilde{C}) \rightarrow L$  of  $p_L$  is an injection.

Let  $\Delta = \bigcup_a \Delta_a$  denote the image of  $\mathbf{P}(\tilde{C}) = \bigcup_a \mathbf{P}(\tilde{C}_a)$  by the projection  $X \times_{\mathbf{P}} Q \rightarrow \mathbf{P}^\vee$ . By [S, Corollary 3.21], there exists a closed subset  $\Delta' \subset \Delta$  such that  $\Delta' \subset \mathbf{P}^\vee$  is of codimension  $\geq 2$  and that  $\mathbf{P}(\tilde{C}) \rightarrow \Delta$  is finite radicial outside of  $\Delta'$ . Further, the image  $\Delta_a \subset \mathbf{P}^\vee$  of  $\mathbf{P}(\tilde{C}_a)$  is a divisor for each irreducible component  $C_a$  of  $C$  and  $\Delta_a \cap \Delta_b \subset \Delta'$  for every pair of irreducible components  $C_a \neq C_b$  of  $C$ . Hence, the lines  $L \subset \mathbf{P}^\vee$  satisfying the conditions (3) and (4) form a dense open subset  $U_2 \subset U_0 \subset \mathbf{G}$ .

We set  $U = U_1 \cap U_2$ . Let  $L \subset \mathbf{P}^\vee$  be a line in  $U$ . As we have seen, the line  $L$  satisfies the conditions (1)–(4) above. The line  $L$  also satisfies the condition (5) by Lemma 2.2.2 (2) $\Rightarrow$ (3). For each irreducible component  $C_a$  of  $C$ , since the image  $\Delta_a \subset \mathbf{P}^\vee$  is a divisor, the intersection  $\Delta_a \cap L$  is non-empty. Since  $\Delta_a \cap L$  is the image of  $X_L \cap \mathbf{P}(\tilde{C}_a)$ , the condition (6) is satisfied. Since  $\Delta'$  does not meet  $L$  by the construction of  $U$ , the condition (7) is satisfied.  $\square$

### 3 Euler-Poincaré characteristics determine the characteristic cycle

We give a sufficient condition for constructible complexes to have the same characteristic cycles. This will be used in the proof of Theorem 0.1 in the next section.

**Definition 3.1.** Let  $X$  be a scheme of finite type over a field  $k$  and let  $\bar{k}$  be an algebraic closure of  $k$ . Let  $\Lambda$  and  $\Lambda'$  be finite fields of characteristics different from that of  $k$  and let  $\mathcal{F}$  and  $\mathcal{F}'$  be constructible complexes of  $\Lambda$ -modules and of  $\Lambda'$ -modules respectively. We say that  $\mathcal{F}$  and  $\mathcal{F}'$  have *universally the same Euler-Poincaré characteristics* if for every separated scheme  $Z$  of finite type over  $k$  and for every morphism  $g: Z \rightarrow X$  over  $k$ , we have  $\chi_c(Z_{\bar{k}}, g^* \mathcal{F}) = \chi_c(Z_{\bar{k}}, g^* \mathcal{F}')$ .

If  $k$  is of characteristic 0, the condition is equivalent to that  $\dim \mathcal{F}_{\bar{x}} = \dim \mathcal{F}'_{\bar{x}}$  for every geometric point  $\bar{x}$  of  $X$ .

**Lemma 3.2.** Let  $X$  be a scheme of finite type over a field  $k$ . Let  $\Lambda$  and  $\Lambda'$  be finite fields of characteristics different from that of  $k$  and let  $\mathcal{F}$  and  $\mathcal{F}'$  be constructible complexes

of  $\Lambda$ -modules and of  $\Lambda'$ -modules respectively with universally the same Euler-Poincaré characteristics.

1. Let  $h: W \rightarrow X$  be a morphism of schemes of finite type over  $k$ . Then  $h^*\mathcal{F}$  and  $h^*\mathcal{F}'$  have universally the same Euler-Poincaré characteristics.

2. Let  $f: X \rightarrow Y$  be a separated morphism of schemes of finite type over  $k$ . Then,  $Rf_!\mathcal{F}$  and  $Rf_!\mathcal{F}'$  have universally the same Euler-Poincaré characteristics.

*Proof.* We may assume  $k$  is algebraically closed. Let  $Z$  be a separated scheme of finite type over  $k$ .

1. Let  $g: Z \rightarrow W$  be a morphism over  $k$ . Then, we have  $\chi_c(Z, g^*h^*\mathcal{F}) = \chi_c(Z, g^*h^*\mathcal{F}')$  and the assertion follows.

2. Let  $g: Z \rightarrow Y$  be a morphism over  $k$  and let  $g': Z' \rightarrow X$  be the base change. Since the proper base change theorem implies the equalities  $\chi_c(Z, g^*Rf_!\mathcal{F}) = \chi_c(Z', g'^*\mathcal{F}) = \chi_c(Z', g'^*\mathcal{F}') = \chi_c(Z, g^*Rf_!\mathcal{F}')$ , the assertion follows.  $\square$

We recall the definition of the relative Grothendieck group  $K_0(\mathcal{V}_X)$  [L, 2]. Let  $\mathcal{V}_X$  denote the category consisting of morphisms  $g: Z \rightarrow X$  of schemes over  $k$  where  $Z$  is a separated scheme of finite type over  $k$ . The Grothendieck group  $K_0(\mathcal{V}_X)$  is the quotient of free abelian group generated by isomorphisms classes  $[g: Z \rightarrow X]$  of objects of  $\mathcal{V}_X$ , divided by the relations  $[g: Z \rightarrow X] - [g_V: V \rightarrow X] = [g_W: W \rightarrow X]$  for closed subschemes  $V \subset Z$  and the complement  $W = Z - V$  where  $g_V$  and  $g_W$  denote the restrictions of  $g$ .

Let  $\tilde{F}(X) = \text{Hom}(K_0(\mathcal{V}_X), \mathbf{Z})$  denote the dual abelian group. For a morphism  $h: W \rightarrow X$  of schemes of finite type over  $k$ , the functor  $h_*: \mathcal{V}_W \rightarrow \mathcal{V}_X$  sending  $[g: Z \rightarrow W]$  to  $[h \circ g: Z \rightarrow X]$  induces a morphism  $h_*: K_0(\mathcal{V}_W) \rightarrow K_0(\mathcal{V}_X)$  and its dual  $h^*: \tilde{F}(X) \rightarrow \tilde{F}(W)$ . For a separated morphism  $f: X \rightarrow Y$  of schemes of finite type over  $k$ , the functor  $f^*: \mathcal{V}_Y \rightarrow \mathcal{V}_X$  sending  $[g: Z \rightarrow Y]$  to  $[g': Z \times_Y X \rightarrow X]$  induces a morphism  $f^*: K_0(\mathcal{V}_Y) \rightarrow K_0(\mathcal{V}_X)$  and its dual  $f_!: \tilde{F}(X) \rightarrow \tilde{F}(Y)$ .

Let  $K(X, \Lambda)$  denote the Grothendieck group of constructible sheaves of  $\Lambda$ -modules on  $X$ . Then, the pairing  $K(X, \Lambda) \times K_0(\mathcal{V}_X) \rightarrow \mathbf{Z}$  defined by  $(\mathcal{F}, [g: Z \rightarrow X]) \mapsto \chi_c(Z_k, g^*\mathcal{F})$  induces a morphism

$$(3.1) \quad K(X, \Lambda) \rightarrow \tilde{F}(X).$$

The proof of Lemma 3.2.1 shows that, for a morphism  $h: W \rightarrow X$  of schemes of finite type over  $k$ , we have a commutative diagram

$$(3.2) \quad \begin{array}{ccc} K(X, \Lambda) & \longrightarrow & \tilde{F}(X) \\ h^* \downarrow & & \downarrow h^* \\ K(W, \Lambda) & \longrightarrow & \tilde{F}(W). \end{array}$$

The proof of Lemma 3.2.2 also shows that, for a separated morphism  $f: X \rightarrow Y$  of schemes of finite type over  $k$ , we have a commutative diagram

$$(3.3) \quad \begin{array}{ccc} K(X, \Lambda) & \longrightarrow & \tilde{F}(X) \\ f_! \downarrow & & \downarrow f_! \\ K(Y, \Lambda) & \longrightarrow & \tilde{F}(Y). \end{array}$$



Let  $C$  be a connected smooth curve over a perfect field  $k$  and  $\mathcal{F}$  be a constructible complex of  $\Lambda$ -modules on  $C$ . Let  $\text{rank } \mathcal{F}$  denote the alternating sum  $\sum_q (-1)^q \text{rank } \mathcal{H}^q \mathcal{F}|_U$  on a dense open subscheme  $U \subset C$  where the restrictions  $\mathcal{H}^q \mathcal{F}|_U$  of cohomology sheaves are locally constant. For a closed point  $v \in C$ , the Artin conductor is defined by

$$(3.4) \quad a_v \mathcal{F} = \text{rank } \mathcal{F} - \dim \mathcal{F}_{\bar{v}} + \text{Sw}_v \mathcal{F}$$

where  $\bar{v}$  denotes a geometric point above  $v$  and  $\text{Sw}_v$  denotes the alternating sum of the Swan conductor.

**Lemma 3.3.** *Let  $X$  be a scheme of finite type over a perfect field  $k$ . Let  $\Lambda$  and  $\Lambda'$  be finite fields of characteristics different from that of  $k$  and let  $\mathcal{F}$  and  $\mathcal{F}'$  be constructible complexes of  $\Lambda$ -modules and of  $\Lambda'$ -modules respectively with universally the same Euler-Poincaré characteristics.*

*Let  $\bar{C}$  be a proper smooth curve over  $k$ , let  $j: C \rightarrow \bar{C}$  be the open immersion of a dense open subscheme and  $g: C \rightarrow X$  be a morphism over  $k$ . Then, for a closed point  $v \in \bar{C}$ , we have an equality of the Artin conductors*

$$(3.5) \quad a_v j_! g^* \mathcal{F} = a_v j_! g^* \mathcal{F}'.$$

*Proof.* We may assume  $k$  is algebraically closed. Since the dimensions of fibers of  $j_! g^* \mathcal{F}$  and  $j_! g^* \mathcal{F}'$  at each points are equal, it suffices to show the equality of the Swan conductors:  $\text{Sw}_v g^* \mathcal{F} = \text{Sw}_v g^* \mathcal{F}'$ . We may further assume  $X = C$  by Lemma 3.2,  $v \notin X$  and that  $\mathcal{F}$  and  $\mathcal{F}'$  are locally constant.

Let  $\bar{X}$  be a smooth compactification of  $X$ . By approximation, there exists a finite morphism  $\bar{Z} \rightarrow \bar{X}$  of proper smooth curves étale at  $v$  such that the pull-backs of  $\mathcal{F}$  and  $\mathcal{F}'$  on  $Z = X \times_{\bar{X}} \bar{Z}$  are unramified along  $\bar{Z} - (\bar{Z} \times_{\bar{X}} (X \cup \{v\}))$ . Then, by the Grothendieck-Ogg-Shafarevich formula [SGA5, Théorème 7.1], we have  $[Z : X] \cdot \text{Sw}_v \mathcal{F} = \text{rank } \mathcal{F} \cdot \chi_c(Z) - \chi_c(Z, \mathcal{F})$  and similarly for  $\mathcal{F}'$ . Thus the assertion follows.  $\square$

**Proposition 3.4.** *Let  $X$  be a smooth scheme over a perfect field  $k$  and let  $\bar{k}$  be an algebraic closure of  $k$ . Let  $\Lambda$  and  $\Lambda'$  be finite fields of characteristics different from that of  $k$  and let  $\mathcal{F}$  and  $\mathcal{F}'$  be constructible complexes of  $\Lambda$ -modules and of  $\Lambda'$ -modules respectively.*

*If  $\mathcal{F}$  and  $\mathcal{F}'$  have universally the same Euler-Poincaré characteristics, we have*

$$(3.6) \quad CC\mathcal{F} = CC\mathcal{F}'.$$

*Proof.* We may assume that  $k$  is algebraically closed. Since the question is local, we may assume  $X$  is affine. We take an immersion  $i: X \rightarrow \mathbf{A}^n \subset \mathbf{P}^n$ . Since  $Ri_! \mathcal{F}$  and  $Ri_! \mathcal{F}'$  have universally the same Euler-Poincaré characteristics by Lemma 3.2.2, we may assume  $X$  is projective.

We take an immersion  $X \rightarrow \mathbf{P}$  such that the pull-back  $\mathcal{L}$  of  $\mathcal{O}(1)$  satisfies the conditions (E) and (C). Let  $C = SS\mathcal{F} \cup SS\mathcal{F}'$  be the union of the singular supports. Let  $C_a$  be an irreducible component and we show that the coefficients  $m_a$  and  $m'_a$  of  $C_a$  in  $CC\mathcal{F}$  and  $CC\mathcal{F}'$  are equal.

Let  $\pi_L: X_L \rightarrow X$  and  $p_L: X_L \rightarrow L$  be morphisms satisfying the conditions (1)–(7) in Lemma 2.3. Let  $x \in X_L \cap \mathbf{P}(\tilde{C}_a)$  be an isolated characteristic point of  $p_L: X_L \rightarrow L$  and  $y = p_L(x)$  be the image. By the Milnor formula, we have

$$-a_y R p_{L*} \pi_L^* \mathcal{F} = -\dim \text{tot} \phi_x(\pi_L^* \mathcal{F}, p_L) = (CC\mathcal{F}, dp_L)_{T^* X, x} = m_a \cdot (C_a, dp_L)_{T^* X, x}$$

and similarly for  $\mathcal{F}'$ . Since  $a_y R p_{L*} \pi_L^* \mathcal{F} = a_y R p_{L*} \pi_L^* \mathcal{F}'$  by Lemma 3.2 and Lemma 3.3, we have  $m_a \cdot (C_a, dp_L)_{T^*X, x} = m'_a \cdot (C_a, dp_L)_{T^*X, x}$ . Since  $(C_a, dp_L)_{T^*X, x} \neq 0$ , we obtain  $m_a = m'_a$  as required.  $\square$

Let  $X$  be a scheme of finite type over  $k$ . Assume that there exists a closed immersion  $X \rightarrow M$  to a smooth scheme over  $k$  and let  $cc_X: K_0(X, \Lambda) \rightarrow CH_\bullet(X)$  denote the morphism defined by characteristic classes [S, Definition 6.7]. Let  $K_0(X, \Lambda)_0 \subset K_0(X, \Lambda)$  denote the kernel of the morphism (3.1). Then, Proposition 3.4 implies that the morphism  $cc_X: K_0(X, \Lambda) \rightarrow CH_\bullet(X)$  factors through the quotient  $K_0(X, \Lambda)/K_0(X, \Lambda)_0$ .

## 4 Brauer traces and representations of $p$ -groups

We briefly recall the definition of the Brauer trace of a semi-simple automorphism of a vector space over a finite field. Let  $\Lambda$  be a finite field of characteristic  $\ell$  and  $E = W(\Lambda)[\frac{1}{\ell}]$  be the fraction field of the ring of Witt vectors. Let  $M$  be a  $\Lambda$ -vector space of finite dimension  $n$  and let  $\sigma$  be an automorphism of  $M$  of order prime to  $\ell$ . Decompose the characteristic polynomial  $\Phi(T) = \det(T - \sigma : M)$  as  $\Phi(T) = \prod_{i=1}^n (T - a_i)$  and let  $\tilde{a}_i$  be the unique lifting of  $a_i$  as a root of 1 of order prime to  $\ell$  in a finite unramified extension of  $E$ . Then, the Brauer trace  $\text{Tr}^{Br}(\sigma, M) \in E$  is defined by

$$(4.1) \quad \text{Tr}^{Br}(\sigma, M) = \sum_{i=1}^n \tilde{a}_i.$$

**Lemma 4.1.** *Let  $\Lambda$  be a finite field,  $M$  be a  $\Lambda$ -vector space of finite dimension and let  $\sigma$  be an automorphism of  $M$  of order a power of prime  $p$  invertible in  $\Lambda$ . Then, for a subfield  $E$  of the fraction field of  $W(\Lambda)$  of finite degree over  $\mathbf{Q}$  containing  $\text{Tr}^{Br}(\sigma, M)$ , we have*

$$(4.2) \quad \frac{1}{[E : \mathbf{Q}]} \text{Tr}_{E/\mathbf{Q}} \text{Tr}^{Br}(\sigma, M) = \frac{1}{p-1} (p \cdot \dim M^\sigma - \dim M^{\sigma^p}).$$

*Proof.* By devissage, we may assume  $M$  is of dimension 1 and  $E$  is generated by the lifting of the eigenvalue  $\zeta$  of  $\sigma$ . If  $\zeta = 1$ , the equality (4.2) is  $1 = \frac{1}{p-1}(p-1)$ . If  $\zeta$  is of order  $p$ , it is  $\frac{1}{p-1}(-1) = \frac{1}{p-1}(0-1)$ . If otherwise, it is  $0 = 0$ .  $\square$

We study representations of a  $p$ -group. Let  $p$  be a prime number,  $G$  be a finite  $p$ -group of order  $p^n$  and  $\Lambda$  be a finite field of characteristic different from  $p$ . Let  $K(G, \Lambda)$  (resp.  $K(G, \mathbf{Q})$ ) be the Grothendieck group of  $\Lambda$ -representations (resp.  $\mathbf{Q}$ -representations) of  $G$ . The subfield  $E$  of the fraction field of the ring of Witt vectors  $W(\Lambda)$  generated over  $\mathbf{Q}$  by the values of Brauer traces  $\text{Tr}^{Br}(\sigma : M)$  for  $\sigma \in G$  and  $\Lambda$ -representations  $M$  of  $G$  is a subfield of  $\mathbf{Q}(\zeta_{p^n})$ .

Let  $\text{Cent}(G, E)$  (resp.  $\text{Cent}(G, \mathbf{Q})$ ) denote the space of central functions. Then, the Brauer traces define an injection  $\text{Tr}^{Br}: K(G, \Lambda) \rightarrow \text{Cent}(G, E)$ . The image of the injection  $\text{Tr}: K(G, \mathbf{Q}) \otimes \mathbf{Q} \rightarrow \text{Cent}(G, \mathbf{Q})$  consists of central functions  $f: G \rightarrow \mathbf{Q}$  satisfying  $f(\sigma) = f(\tau)$  for  $\sigma, \tau \in G$  satisfying  $\langle \sigma \rangle = \langle \tau \rangle$  by [Se, 13.1 Théorème 30].

Let  $A$  be the center of the group algebra  $\mathbf{Z}[G]$ . Since  $\mathbf{Q}[G]$  is semi-simple, we have a canonical isomorphism  $K(G, \mathbf{Q}) \rightarrow \Gamma(\text{Spec } A \otimes \mathbf{Q}, \mathbf{Z})$ . Further, the center  $A/\ell A$  is reduced for every prime  $\ell \neq p$  since  $\mathbf{F}_\ell[G]$  is semi-simple and the ring  $A[\frac{1}{p}]$  is finite étale over  $\mathbf{Z}[\frac{1}{p}]$ . Hence the restriction  $\Gamma(\text{Spec } A[\frac{1}{p}], \mathbf{Z}) \rightarrow \Gamma(\text{Spec } A \otimes \mathbf{Q}, \mathbf{Z})$  is an isomorphism. Let  $A[\frac{1}{p}] =$

$\prod_{i \in I} A_i$  be the decomposition into product of integral domains and  $e_i \in A[\frac{1}{p}] \subset \mathbf{Z}[G][\frac{1}{p}]$  be the corresponding idempotents. Let  $V_i$  be an irreducible  $\mathbf{Q}$ -representation of  $G$  satisfying  $e_i V_i = V_i$  and let  $\chi_i: G \rightarrow \mathbf{Z}$  be the character. Then, the characters  $\chi_i$  form an orthogonal basis of the image of  $K(G, \mathbf{Q}) \otimes \mathbf{Q} \rightarrow \text{Cent}(G, \mathbf{Q})$  with respect to the inner product [Se, 2.2 Remarque]. The authors learned the following fact from Beilinson.

**Lemma 4.2.** *Let  $p$  be a prime number and  $G$  be a finite group of order  $p^n$ . Let  $\Lambda$  be a finite field of characteristic  $\neq p$  and we consider morphisms*

$$(4.3) \quad \begin{array}{ccc} K(G, \Lambda) & \xrightarrow{\text{Tr}^{Br}} & \text{Cent}(G, E) \\ & & \downarrow \frac{1}{[E:\mathbf{Q}]} \text{Tr}_{E/\mathbf{Q}} \\ K(G, \mathbf{Q}) & \xrightarrow{\text{Tr}} & \text{Cent}(G, \mathbf{Q}) \end{array}$$

in the notation above. Let  $M$  be a  $\Lambda$ -representation of  $G$  and for  $\sigma \in G$ , let  $M^\sigma$  denote the  $\sigma$ -fixed part.

1. The image  $s_M \in \text{Cent}(G, \mathbf{Q})$  of the class  $[M] \in K(G, \Lambda)$  lies in the image of the injection  $\text{Tr}: K(G, \mathbf{Q}) \otimes \mathbf{Q} \rightarrow \text{Cent}(G, \mathbf{Q})$  and is given by

$$(4.4) \quad s_M(\sigma) = \frac{1}{p-1} (p \cdot \dim M^\sigma - \dim M^{\sigma^p}).$$

2. Let  $A$  be the center of the group algebra  $\mathbf{Z}[G]$ . Then, we have

$$(4.5) \quad s_M = \sum_i \frac{\dim_\Lambda e_i M}{\dim V_i} \chi_i$$

where  $e_i \in A[\frac{1}{p}]$  runs through primitive idempotents. In other words, under the identification  $\Gamma(\text{Spec } A[\frac{1}{p}], \mathbf{Q}) = K(G, \mathbf{Q}) \otimes \mathbf{Q} \subset \text{Cent}(G, \mathbf{Q})$ , the locally constant function on  $\text{Spec } A[\frac{1}{p}]$  corresponding to  $s_M$  takes values  $\dim_\Lambda e_i M / \dim V_i$  on the components  $\text{Spec } A[\frac{1}{p}]_{e_i}$ .

*Proof.* 1. By Lemma 4.1, the function  $s_M: G \rightarrow \mathbf{Q}$  is given by (4.4). By [Se, 13.1 Théorème 30] and (4.4), the function  $s_M$  lies in the image of  $K(G, \mathbf{Q}) \otimes \mathbf{Q}$ .

2. We may assume  $M = e_i M$ . Then  $s_M$  is orthogonal to  $\chi_j$  for  $j \neq i$  and hence  $s_M$  is a multiple of  $\chi_i$ . Since  $s_M(1) = \dim M$  and  $\chi_i(1) = \dim V_i$ , the assertion follows.  $\square$

## 5 Same wild ramification

Let  $\overline{X}$  be a normal scheme of finite type over a field  $k$  and  $X \subset \overline{X}$  be a dense open subscheme. Let  $G$  be a finite group and  $W \rightarrow X$  be a  $G$ -torsor. The normalization  $\overline{W} \rightarrow \overline{X}$  in  $W$  carries a natural action of  $G$ . For a geometric point  $\bar{x}$  of  $\overline{X}$ , the stabilizer  $I \subset G$  of a geometric point  $\bar{w}$  of  $\overline{W}$  above  $\bar{x}$  is called an inertia subgroup at  $\bar{x}$ .

**Definition 5.1.** Let  $X$  be a scheme of finite type over a field  $k$ . Let  $\Lambda$  and  $\Lambda'$  be finite fields of characteristic invertible in  $k$ . Let  $p \geq 1$  denote the characteristic of  $k$  if  $k$  is of characteristic  $\neq 0$  and set  $p = 1$  if  $k$  is of characteristic 0.

1. Assume that  $X$  is normal and separated. Let  $\mathcal{F}$  and  $\mathcal{F}'$  be locally constant constructible sheaves of  $\Lambda$ -modules and of  $\Lambda'$ -modules on  $X$  respectively. We say that  $\mathcal{F}$  and  $\mathcal{F}'$  have the *same wild ramification* if the following condition is satisfied:

There exists a proper normal scheme  $\bar{X}$  over  $k$  containing  $X$  as a dense open subscheme such that for every geometric point  $\bar{x}$  of  $\bar{X}$ , the following condition is satisfied:

(W) Let  $G$  be a finite quotient group of the inertia group  $I_{\bar{x}} = \pi_1(\bar{X}_{(\bar{x})} \times_{\bar{X}} X, \bar{t})$  with respect to a base point  $\bar{t}$  such that the pull-backs to  $\bar{X}_{(\bar{x})} \times_{\bar{X}} X$  of  $\mathcal{F}$  and  $\mathcal{F}'$  correspond to  $G$ -modules  $M$  and  $M'$  respectively. Then, for every element  $\sigma \in G$  of  $p$ -power order, we have an equality of the dimensions of the  $\sigma$ -fixed parts:

$$(5.1) \quad \dim M^\sigma = \dim M'^\sigma.$$

2. Let  $\mathcal{F}$  and  $\mathcal{F}'$  be constructible complexes of  $\Lambda$ -modules and of  $\Lambda'$ -modules on  $X$  respectively. We say that  $\mathcal{F}$  and  $\mathcal{F}'$  have the *same wild ramification* if the following condition is satisfied:

There exists a finite partition  $X = \coprod_{i \in I} X_i$  by locally closed normal and separated subschemes such that for every  $q$  and for every  $i$ , the restrictions  $\mathcal{H}^q(\mathcal{F})|_{X_i}$  and  $\mathcal{H}^q(\mathcal{F}')|_{X_i}$  are locally constant constructible and have the same wild ramification in the sense defined in 1.

Note that  $\Lambda$  and  $\Lambda'$  are allowed to have the same characteristic. Since the function  $s_M$  (4.4) is determined by the function  $\sigma \mapsto \dim M^\sigma$  and vice versa, the equality (5.1) is equivalent to

$$(5.2) \quad \dim e_i M = \dim e_i M'$$

for every primitive idempotent  $e_i$  of the center  $A[\frac{1}{p}]$  of the group algebra  $\mathbf{Z}[P][\frac{1}{p}]$  of a  $p$ -Sylow subgroup  $P$  of  $G$  by Lemma 4.2.

**Lemma 5.2.** *Let  $X$  be a scheme of finite type over a field  $k$ . Let  $\Lambda$  and  $\Lambda'$  be finite fields of characteristic invertible in  $k$ . Let  $\mathcal{F}$  and  $\mathcal{F}'$  be constructible complexes of  $\Lambda$ -modules and of  $\Lambda'$ -modules on  $X$ . Let  $h: W \rightarrow X$  be a morphism of schemes of finite type over  $k$ .*

*If  $\mathcal{F}$  and  $\mathcal{F}'$  have the same wild ramification, then the pull-backs  $h^*\mathcal{F}$  and  $h^*\mathcal{F}'$  also have the same wild ramification.*

*Proof.* By devissage, we may assume that  $X$  is normal and separated and that  $\mathcal{F}$  and  $\mathcal{F}'$  are locally constant constructible sheaves satisfying the condition in Definition 5.1.1. Let  $\bar{X}$  be a proper normal scheme containing  $X$  as a dense open subscheme satisfying the condition in Definition 5.1.1. Further by devissage, we may assume  $W$  is normal and affine. Let  $W'$  be a projective normal scheme over  $k$  containing  $W$  as a dense open subscheme. Let  $\bar{W}$  be the normalization of the closure of the image of  $W$  in  $W' \times \bar{X}$ . Then,  $h^*\mathcal{F}$  and  $h^*\mathcal{F}'$  satisfy the condition in Definition 5.1.1.  $\square$

We will deduce Proposition 0.2 from the following lemma.

**Lemma 5.3** (cf. [I, Théorème 2.1]). *Let  $\bar{X}$  be a proper normal scheme over an algebraically closed field  $k$  of characteristic  $p > 0$  and let  $X \subset \bar{X}$  be a dense open subscheme. Let  $G$  a finite group and  $W \rightarrow X$  be a  $G$ -torsor.*

1. ([DL, 3.3]) *The trace  $\text{Tr}(\sigma : H_c^*(W, \mathbf{Z}_\ell))$  is an integer independent of  $\ell \neq p$ .*

2. *Let  $\Lambda$  be a finite fields of characteristic invertible in  $k$  and let  $\mathcal{F}$  be a locally constant constructible sheaf of  $\Lambda$ -modules on  $X$  such that the pull-back to  $W$  of  $\mathcal{F}$  is constant. Let*

$M$  be the  $G$ -module corresponding to  $\mathcal{F}$ . Let  $S \subset G$  be the subset consisting of elements of  $p$ -power order contained in the inertia groups at a geometric point of  $\bar{X}$ . Then, we have

$$(5.3) \quad \chi_c(X, \mathcal{F}) = \frac{1}{|G|} \sum_{\sigma \in S} \text{Tr}(\sigma : H_c^*(W, \mathbf{Z}_\ell)) \cdot \frac{1}{p-1} (p \cdot \dim M^\sigma - \dim M^{\sigma^p}).$$

*Proof.* The proof is based on that of [I, Théorème 2.1]. By the proof of [I, Lemme 2.5], we have  $\text{Tr}(\sigma : H_c^*(W, \mathbf{Z}_\ell)) = 0$  for  $\sigma \notin S \subset G$  and we have

$$\chi_c(X, \mathcal{F}) = \frac{1}{|G|} \sum_{\sigma \in S} \text{Tr}(\sigma : H_c^*(W, \mathbf{Z}_\ell)) \cdot \text{Tr}^{Br}(\sigma, M).$$

Since  $\chi_c(X, \mathcal{F})$  is an integer and  $\text{Tr}(\sigma : H_c^*(W, \mathbf{Z}_\ell))$  are integers for  $\sigma \in G$ , by taking a subfield  $E$  of the fraction field of  $W(\Lambda)$  of finite degree over  $\mathbf{Q}$  containing  $\text{Tr}^{Br}(\sigma, M)$  for  $\sigma \in S$  and applying Lemma 4.1, we obtain (5.3).  $\square$

*Proof of Proposition 0.2.* We may assume that  $k$  is algebraically closed. By Lemma 5.2 and devissage, we may assume that  $X$  is normal and affine and that  $\mathcal{F}$  and  $\mathcal{F}'$  are locally constant constructible sheaves satisfying the condition in Definition 5.1.1. If  $k$  is of characteristic 0, we have  $\chi_c(X_{\bar{k}}, \mathcal{F}) = \text{rank } \mathcal{F} \cdot \chi_c(X_{\bar{k}}, \Lambda)$  by [I, Théorème 2.1] and similarly for  $\mathcal{F}'$  and the assertion follows.

Assume  $k$  is of characteristic  $p > 0$  and let  $\bar{X}$  be a proper normal scheme containing  $X$  as a dense open subscheme and satisfying the condition in loc. cit. Then, Lemma 5.3.2 implies  $\chi_c(X_{\bar{k}}, \mathcal{F}) = \chi_c(X_{\bar{k}}, \mathcal{F}')$ .  $\square$

**Corollary 5.4.** *Let  $X$  be a scheme of finite type over a field  $k$ . Let  $\Lambda$  and  $\Lambda'$  be finite fields of characteristic invertible in  $k$ . Let  $\mathcal{F}$  and  $\mathcal{F}'$  be constructible complexes of  $\Lambda$ -modules and of  $\Lambda'$ -modules on  $X$  with the same wild ramification. Then,  $\mathcal{F}$  and  $\mathcal{F}'$  have universally the same Euler-Poincaré characteristics.*

*Proof.* By Lemma 5.2, it follows from Proposition 0.2.  $\square$

*Proof of Theorem 0.1.* By Corollary 5.4, it follows from Proposition 3.4.  $\square$

The definition of the property having the same wild ramification is different from that studied in [V, Definition 2.3.1]. It may be also interesting to consider a generalization to algebraic spaces as in [IZ, Section 5].

In this note, we formulated the independence of  $\ell$  in terms of wild inertia. At least if  $k$  is finite, one can replace this by the condition on the traces of Frobenius as in [F], [Z] by using the Chebotarev density theorem. W. Zheng further suggested to consider the subgroup of the Grothendieck group of  $E$ -compatible systems of constructible complexes [Z, Définition 1.14] consisting of classes of virtually trivial wild ramification as in [V, Definition 2.3.1] and extend the results of [V] to this framework. L. Illusie suggested that one can also prove a statement analogous to Theorem 0.1 for the singular support. The authors thank them for the remarks.

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